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BUOYANT PLUME RISE IN ATMOSPHERIC INVERSIONS

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Abstract—The rise of a buoyant plume into atmospheric inversions is investigated analytically for a range of inversion rates. The analysis assumes gaussian distribution of velocity and temperature. The results show that maximum plume rise decreases with inversion intensity, and the ratio of maximum rise to the height at which zero buoyancy occurs is a function of the densimetric froude number and the inversion intensity.

NOMENCLATURE

a,	5.729 plume profile parameter;
$b(z)$,	plume radius;
c,	atmospheric inversion parameter [K];
Fr,	$u/gb(0)B$ initial densimetric Froude number;
Fr_ϕ,	local densimetric Froude number;
g,	gravitational constant;
I_{ij},	similarity variable, $i = 1, 2, 3, j = 1, 2$;
$P(z)$,	atmospheric pressure;
g_r,	radial heat transfer;
r,	radial coordinate of plume;
R_g,	gas constant;
R,	$b(z)/b(0)$ dimensional radius;
S,	atmospheric inversion parameter;
T,	temperature;
$u(r, z)$,	plume axial velocity;
u_0,	$u(r = 0, z)$;
u_l,	$u(0, 0)$;
U,	$u(0, z)/u_l$ dimensionless plume velocity;
$V(r, z)$,	plume radial velocity;
V_e,	$v(b, z)$;
Ve,	$v(b, z)/u_l$ dimensionless entrainment velocity;
Z,	axial distance along plume;
Z_i,	dimensionless density variable, $i = 1, 2, 3$.

Greek symbols

α,	entrainment parameter;
α_j,	components of entrainment parameter $j = 1, 2, 3$;
β,	$\rho_l - \rho_{lp}$ density difference;
ρ,	density;
ρ_l,	air reference density;
τ_{rz},	turbulent shear stress distribution;
ξ,	$Z/b(0)$ dimensionless distance;
ϕ,	$\rho_e - \rho(0, z)$ or $(\rho_e - \rho(0, z)/B$;
η,	$r/b(z)$ dimensionless radial variable.

Subscripts

l,	ground level;
e,	ambient condition;
p,	plume.

1. INTRODUCTION

INTEREST in the maximum plume rise, and buoyant plume entrainment has increased recently as the atmosphere is being considered as the ultimate heat sink for process, thermal and nuclear power plants. The heights to which plumes rise determine the extent and range of effluent dilution. In the case of liquid laden plumes, the dispersion of plumes determines the extent of misting and reprecipitation of moisture to ground level. The rise of gaseous plumes from dry cooling tower stacks have been studied as a model for understanding the rise and spread of moisture laden plumes of wet cooling towers. On account of the tendency for recondensation and misting in a wet plume, any reduction in its overall rise is undesirable from an environmental point of view. Since entrainment influences the thermal energy of the plume, it will be instructive to investigate the entrainment characteristic of a dry plume, and to observe the effect of inversion intensity on the overall plume rise.

Morton, Taylor and Turner [1] proposed that the entrainment velocity of a plume is proportional to the axial core velocity $u(r = 0, z)$, with the proportionality variable being constant. Briggs [2], following an extensive literature survey, proposed that the entrainment velocity is proportional to the square root of the axial momentum flux u^2b^2, and the proportionality variable, α, is constant. In the analytical study by Fox [3], it was shown that the parameter α is related as $\alpha = \alpha_1 + \alpha_2/Fr_\phi^2$ in which α_1 and α_2 are constants, and Fr_ϕ is the local densimetric Froude number. Although Fox's model is in agreement with [1], his expression for the densimetric Froude number leads to imaginary numbers in the regions of negative buoyancy. The analysis by Fox established the occurrence of outflow from the plume; extending his result into the negative buoyancy region is questionable since Fr_ϕ^2 is negative there. The differences in the existing models may be identified with the closure of the governing equations. In [1], no closure was necessary, the relationship between entrainment velocity and $u(r = 0, z)$ was assumed explicitly. In [3], the turbulent shear stress was prescribed independently, thus yielding a free parameter that was determined empirically.

The objective of this study is to investigate the rise of a dry buoyant plume subject to the conservation of its mechanical energy. The closure based on the conservation of mechanical energy makes the system of governing equations internally consistent. As in previous studies, it is assumed that the flow is axisymmetric, turbulent and steady in the mean [4, 5]. The plume is nominally compressible, and perfect gas relations are assumed.

2. ANALYSIS

The governing equations are

Continuity

$$\frac{\partial}{\partial r}(\rho r V_r) + \frac{\partial}{\partial z}(\rho r u) = 0 \tag{1}$$

Momentum

$$\frac{\partial}{\partial r}(\rho r V_r u) + \frac{\partial}{\partial z}(\rho r u^2) = (\rho_e - \rho) g r + \frac{\partial}{\partial r}(r \tau_{rz}) \tag{2}$$

Energy

$$\frac{\partial}{\partial r}(\rho r V_r c_p T) + \frac{\partial}{\partial z}(\rho r u c_p T) = \frac{\partial}{\partial r}(r q_r) \tag{3}$$

Closure

$$\frac{\partial}{\partial r}(\rho r V_r u^2) + \frac{\partial}{\partial z}(\rho r u^3) = 2u(\rho_e - \rho) g r + 2u \frac{\partial}{\partial r}(r \tau_{rz}) \tag{4}$$

Equation of State

$$P(r,z) = \rho(r,z) R_g T(r,z) \tag{5}$$

where V_r and u are the radial and axial velocity components in the plume, ρ_e is ambient air density, ρ is the plume density, τ_{rz} the effective shear stress, q_r the turbulent heat transfer in the radial direction, r and z are the radial and axial coordinates of the plume. Equation (4) is the closure equation obtained by multiplying equation (2) with $u(r, z)$ and using the continuity equation to simplify the analysis.

The condition of atmospheric inversion is assumed to be in the form

$$T_e(z) = T_l + cz \tag{6}$$

in which T_l is the ambient air temperature at the plume source, and C is the atmospheric inversion rate, with $c > 0$. From equations (5) and (6) the atmospheric pressure and density variation may be obtained as

$$P_e(z) = P_l\left(1 + \frac{cz}{T_l}\right)^{-g/cR_g} \tag{7}$$

$$\rho_e(z) = \rho_l\left(1 + \frac{cz}{T_l}\right)^{-g/cR_g - 1}. \tag{8}$$

There are thus seven unknown variable ρ, V_r, u, T, P, T_{rz}, and q_r in the governing equations. Equations (1)–(8) supply the seven independent equations necessary for determining the dependent variables. The model is thus well posed, and a solution will now be obtained.

It is reasonable to assume that the velocity and density profiles within the plume are similar. This assumption which is valid for the incompressible flow [4] is expected to hold for a buoyant plume since the maximum temperature defect, occurring at the source, is normally small.

Experimental studies on buoyant plumes confirm the validity of assuming similarity profiles. As with Fox [3] and others [6] it will be assumed that the velocity and density profiles can be written in the form

$$u(r,z) = u_0\, e^{-a\eta^2} \tag{9}$$

$$\rho_e - \rho(r = 0, z) = \phi(z)\, e^{-a\eta^2/\lambda^2} \tag{10}$$

where ϕ is the axial density difference, and λ is a dispersion parameter. Experiments [7] have shown that the temperature or density profiles are not generally confined within the limits of the plume. In the study by Fox [3], it was found that experimental data could be correlated with $0.8 < \lambda < 1.2$. It will be assumed in this study that $\lambda = 1$.

On integrating equations (1)–(4) with respect to r from $r = 0$ to $r = b(z)$, the following equations are obtained.

$$\frac{d}{dz}[b^2 u_0(\rho_e I_{11} - I_{12}\phi)] = \rho_e b V_e \tag{11}$$

$$\frac{d}{dz}[b^2 u_0^2(\rho_e I_{21} - I_{22}\phi)] = \phi b^2 g I_{11} \tag{12}$$

$$\frac{d}{dz}(P_e b^2 u_0) = P_e b V_e / I_{11} \tag{13}$$

$$\frac{d}{dz}[b^2 u_0^3(\rho_e I_{31} - I_{32}\phi)] = A_1 b^2 u_0 \phi g I_{12} + A_2 \rho_e b^2 u_0^3\left(\frac{1}{u_0}\frac{du_0}{dz}\right) + O(\beta, S) \tag{14}$$

where V_e is the entrainment velocity, and the similarity variables $I_{jk}, j = 1, 2, 3, k = 1, 2$ are given by

$$\begin{aligned} I_{11} &= \int_0^1 \eta\, e^{-a\eta^2}\, d\eta \\ I_{12} = I_{21} &= \int_0^1 \eta\, e^{-2a\eta^2}\, d\eta \\ I_{22} = I_{31} &= \int_0^1 \eta\, e^{-3a\eta^2}\, d\eta \\ I_{32} &= \int_0^1 \eta\, e^{-4a\eta^2}\, d\eta \end{aligned} \tag{15}$$

and

$$\begin{aligned} A_1 &= (e^{-a} - e^{-2a})/a + (0.1667 - 0.5\, e^{-a} + 0.5 e^{-2a} - 0.1667\, e^{-3a}) I_{11}/I_{21} \\ A_2 &= 0.1667 - 0.5\, e^{-2a} + 0.333\, e^{-3a}. \end{aligned} \tag{16}$$

The dimensionless form of equations (11)–(14) is

$$\frac{d}{d\xi}(R^2UZ_1) = \rho_e R V_e \tag{17}$$

$$\frac{d}{d\xi}(R^2U^2Z_2) = \frac{I_{11}}{Fr^2}R^2\phi \tag{18}$$

$$\frac{d}{d\xi}(PeR^2U) = PeR^2V_e/I_{11} \tag{19}$$

$$\frac{d}{d\xi}(R^2U^3Z_3) = A_1\phi R^2U/Fr^2 + A_2\rho_e R^2U^3\left(\frac{1}{U}\frac{dU}{d\xi}\right) \tag{20}$$

where the initial Froude number, Fr, and the density variables z_j, $j = 1, 3$ are defined by

$$Fr = u_l/(gb(0)\beta)^{1/2} \tag{21}$$

$$Z_j = \rho_e I_{j1} - \phi\beta I_{j2}, \quad j = 1, 2, 3. \tag{22}$$

The entrainment velocity is eliminated from the continuity and energy equations to obtain

$$\frac{d}{d\xi}(R^2U\phi) = I_{11}SR^2U/(I_{12}\beta). \tag{23}$$

The system of governing equations thus reduces to three simultaneous nonlinear differential equations (18), (20) and (23), and are subject to the initial conditions specified at the plume source

$$\text{at } \xi = \xi_0 \quad \begin{aligned} (R^2U^2Z_2)_0 &= Z_2|_0 \\ (R^2U^3Z_0)_0 &= Z_3|_0 \\ (R^2U\phi)_0 &= 1. \end{aligned} \tag{24}$$

Under Boussinesq approximations, the Z_j, $j = 1, 3$ on the LHS of equations (17)–(20) are generally [1, 3, 6] replaced with $\rho_l I_{ji}$. While this step simplifies the analysis, it leads to the loss of valuable information. Equation (18) can be easily reduced to the form

$$\frac{d}{d\xi}(\rho_e R^2U^2) = \frac{I_{11}}{I_{12}}\frac{R^2\phi}{Fr^2} + \frac{I_{22}}{I_{21}}\beta\phi R^2U^2\left(\frac{1}{U}\frac{dU}{d\xi}\right) + \frac{I_{11}I_{22}}{I_{21}^2}SR^2U \tag{25}$$

which shows the effect of buoyancy and atmospheric inversion on the apparent momentum flow. Under Boussinesq approximation ρ_e is commonly replaced with ρ_l, and the last two terms in equation (25) are dropped. In zero buoyancy flow, in a stably stratified atmosphere, equation (25) preserves the influence of stratification. In addition, it may be noted that the first and second terms on the RHS of equation (25) counteract each other throughout the plume motion. Initially $0 \leqslant \phi \leqslant 1$, and $du/d\xi$ is negative everywhere; in the negative buoyancy zone, $\phi < 0$, the signs of these terms are reversed. For the reasons stated above, the Boussinesq approximation will not be used in this study.

The incompressible limit of the governing equations is obtained when ϕ is set equal to zero, or when the densimetric Froude number becomes large. One infers from these limiting extremes that the plume motion is momentum dominated. The above reasoning allows the variable $1/U\, dU/d\xi$ to be estimated from existing solutions of the incompressible flow.

Further insight into the variation of the entrainment velocity, V_e, is gained by expanding equation (17) and using equation (23) to simplify the result.

$$V_e = I_{11}(\alpha_1 + \alpha_2/Fr_\phi^2 + \alpha_3)RU \tag{26}$$

where

$$\alpha_1 = \left(\frac{I_{22}}{I_{21}}\frac{\beta}{\rho_e} - 1\right)\frac{1}{U}\frac{dU}{d\xi}$$

$$\alpha_2 = \frac{I_{11}}{I_{21}}\operatorname{Sgn}(\phi)$$

$$\alpha_3 = -\left(\frac{I_{22}}{I_{21}} - \frac{I_{12}}{I_{11}}\right)\frac{I_{11}}{I_{12}}\beta S$$

and

$$Fr_\phi = Fr(\rho_e U^2/|\phi|)^{1/2}.$$

Thus the entrainment velocity varies as the square root of the local momentum flow R^2U^2. This is the conclusion reached by Briggs from an extensive literature review. The entrainment parameter may now be written as

$$\alpha = \alpha_1 + \alpha_2/Fr_\phi^2 + \alpha_3. \tag{27}$$

The corresponding expression obtained by Fox can be written as

$$\alpha = 0.0535 + 0.25/Fr_\phi^2. \tag{28}$$

Comparison of equations (27) and (28) suggests α_1 or $(\alpha_1 + \alpha_3)$ is a constant. Equations (26) show that α_1 varies as $1/U\, dU/d\xi$ while α_3 varies as the stratification parameter.

3. CONSTRAINTS ON THE ENTRAINMENT PARAMETER

The components of the entrainment parameter α_j, $j = 1, 2, 3$ show that α is a function of the buoyancy parameter ϕ, the ambient fluid density ρ_e, and the stratification parameter, S. In the initial phase of the plume motion, $0 < \phi < 1$, the influence of the flow parameters Fr, β and S may be investigated for this phase of the plume motion. For large Froude numbers, Fr, or a small density difference $\beta \simeq 0$, equation (26) can be approximated as

$$\alpha = \alpha_1 = -\frac{1}{U}\frac{dU}{d\xi}. \tag{29}$$

These limits correspond to the incompressible flow situation, and Hinze's [4] similarity solution can be used to show that

$$\alpha_1 = \frac{1}{\xi + \xi_0} \qquad \xi \geqslant 0$$

where ξ_0 is the virtual origin. From available analytical and experimental studies, $12 \leqslant \xi_0 \leqslant 16$. Thus at $\xi = 0$, $\alpha_1 = 0.0714$ assuming $\xi_0 = 14$.

The influence of the stratification parameter, S, becomes significant when $\phi < 0$, as α_1 and α_2 no longer augment each other. Under Boussinesq approximation, α_3 is zero as is the density dependent term in the α_1 expression. Equations (25) and (26) are the most general expression for the entrainment parameter. It may be noted in the foregoing that the entrainment velocity V_e, and α become negative only in the negative buoyancy zone. A negative V_e implies outflow from the plume, hence the horizontal spreading out of the plume structure observed in practice.

4. MAXIMUM PLUME RISE ANALYSIS

The assumption of similar profile cannot be expected to hold in the region of plume spread. The plume has residual upward motion even after $V_e \leqslant 0$. At the onset of outflow, the plume radial velocity is zero everywhere. In the region near the plume axis, equation (2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(u^2(0,z)\right) \simeq 2\frac{\rho_e - \rho_p}{\rho_p} g \tag{30}$$

and in dimensionless form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(u^2) \simeq \frac{\rho_e - \rho_p}{\beta} \times \frac{\rho_l}{\rho_p} \times \frac{2}{Fr^2}. \tag{31}$$

Equation (31) is used in calculating the maximum height ξ_{max} at which the plume attains zero upward mobility. The switch from equations (18), (20) and (23) to equation (31) occurs at $V_e = 0$.

5. RESULTS AND CONCLUSIONS

In this study, the numerical calculations were carried out for values of the inversion parameters $C = 1.0$, 0.8, 0.6 and 0.4 degrees rise per unit of axial distance. The height at which the plume begins to spread $V_e = 0$, is shown in Fig. 1. The ratio Z_m/Z_ϕ in Fig. 2 shows the

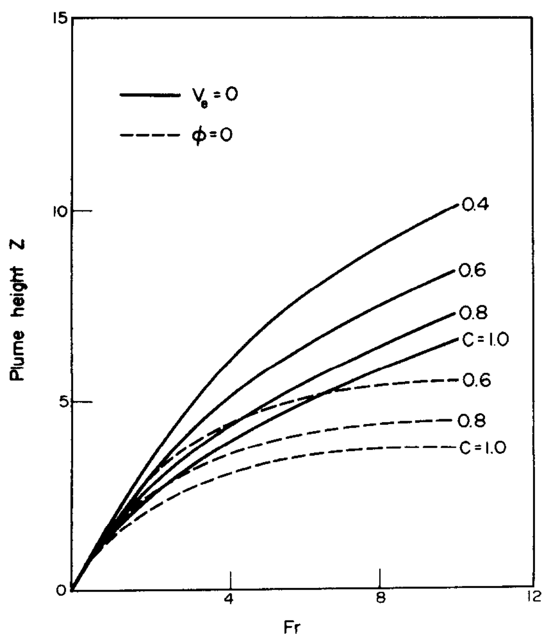

Fig. 1. Influence of inversion intensity on height of plume spread.

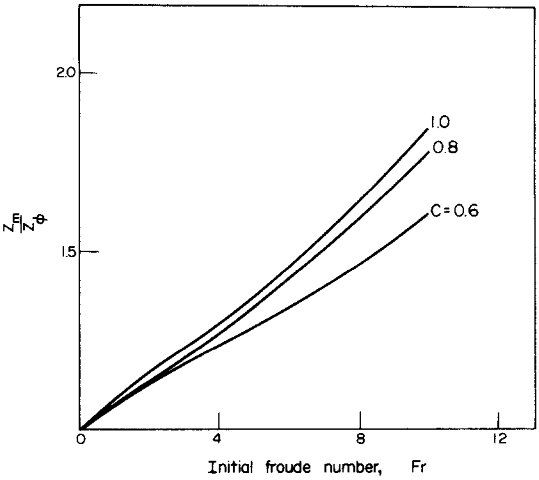

Fig. 2. Dependence of ultimate plume rise on inversion intensity.

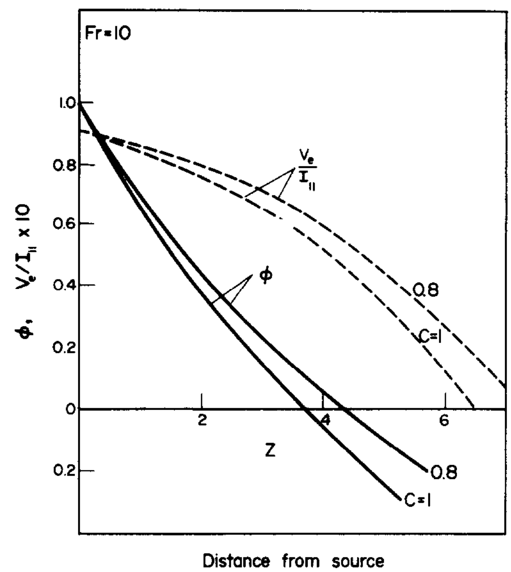

Fig. 3. General characteristics of plume buoyancy and entrainment velocity.

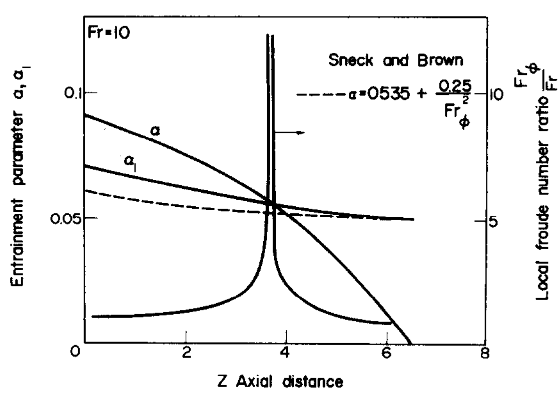

Fig. 4. Characteristics of entrainment and local Froude number.

ultimate plume rise compared to the height at which neutral buoyancy occurs. Figure 3 shows the variation of plume buoyancy and entrainment velocity with plume rise, while the components of plume entrainment parameter are shown in Fig. 4.

The plume rise results obtained in this study are in good agreement with those of Brown and Sneck [6] and Fox [3]. Figure 4 shows that the total entrainment parameter variation obtained in this study differs from the result of Brown and Sneck, shown in Fig. 4. It can be seen that both models predict the same total entrainment of ambient atmospheric fluid. Given the normally assumed small difference between the plume and ambient temperatures, the effect of local variation of entrainment is justifiably small.

REFERENCES

1. B. R. Morton, G. Taylor and J. S. Turner, Turbulent gravitational convection from maintained and instantaneous sources, *Proc. R. Soc.* **A234**, 1–23 (1956).
2. G. A. Briggs, Plume Rise, AEC Critical Review Series, U.S. AEC Div. Tech. Info. (1969).
3. D. G. Fox, Forced plume in a stratified fluid, *J. Geophys. Res.* **75**, (33) (1970).
4. J. O. Hinze, *Turbulence*. McGraw-Hill, New York (1959).
5. S-I. Pai, *Fluid Dynamics of Jets*. Van Nostrand, New York (1954).
6. H. J. Sneck and D. H. Brown, Cooling tower discharge buoyant plume rise through atmospheric inversions, GEC Schenectady, NY, Report No. 71-C-311 (November 1971).
7. S. Corsin and M. S. Uberoi, Further experiments on the flow and heat transfer in a heated turbulent jet, NACA-TM 1865 (April 1949).

ASCENSION D'UN PANACHE THERMIQUE DANS UNE INVERSION ATMOSPHERIQUE

Résumé—On étudie par voie analytique l'élévation d'un panache thermique dans une inversion atmosphérique pour plusieurs taux d'inversion. L'analyse suppose une distribution gaussienne de vitesse et de température. Les résultats montrent que le niveau maximal atteint par le panache décroît avec l'intensité de l'inversion, et que le rapport du niveau maximal à la hauteur pour laquelle les forces d'Archimède s'annulent est une fonction du nombre de Froude et de l'intensité de l'inversion.

DAS AUFSTEIGEN VON AUFTRIEBSSTRÖMUNGEN IN ATMOSPHÄRISCHE INVERSIONSSCHICHTEN

Zusammenfassung—Das Aufsteigen einer Auftriebsströmung in atmosphärische Inversionsschichten wird für verschiedene Inversionsraten analytisch untersucht. Die Studie geht von einer Gauss'schen Verteilung von Geschwindigkeit und Temperatur aus. Die Ergebnisse zeigen, daß die maximale Aufstiegshöhe mit der Inversionsintensität abnimmt, und daß das Verhältnis der maximalen Aufstiegshöhe zur Höhe, bei der der Auftrieb Null wird, eine Funktion der mit den Dichteunterschieden gebildeten Froude–Zahl und der Inversionsintensität ist.

ВОЗНИКНОВЕНИЕ ВСПЛЫВАЮЩИХ СТРУЙ ПРИ АТМОСФЕРНЫХ ИНВЕРСИЯХ

Аннотация — Исследуется аналитически возникновение всплывающей струи при инверсии в атмосфере для различных коэффициентов инверсии. При анализе делается допущение о гауссовом распределении скоростей и температур. Результаты показывают, что максимальный подъем всплывающей струи уменьшается с уменьшением интенсивности инверсии, а отношение максимального подъема к высоте, при которой отсутствует подъемная сила, зависит от денсиметрического числа Фруда и величины инверсии.